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Qualitative behaviour and stability of solutions of discretised nonlinear Volterra integral equations of convolution type

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Abstract

In this work we consider equations of the form:

$$y(t) = g(t) + \int_0^t k(t-s)\varphi(y(s))\,\mathrm{d}s, \qquad t \in \mathbb{R}^+, \tag{†}$$

and their discretised versions (obtained through application of quadrature to (†)) of the form:

$$y_n = g_n + h\sum_{j=0}^{n} w_{n-j}k_{n-j}\varphi_j, \quad \varphi_j = \varphi(y_j), \quad n,j \in \mathbb{N}, \tag{‡}$$

subject to certain conditions on k, φ, g and the weights $\{w_j\}$. One purpose of the paper is to show how the discussion of qualitative behaviour and stability for (†) can be mimicked in a discussion of (‡). We first describe Corduneanu's (1973) discussion of stability for (†), re-presenting his material in a modified form which lends itself to adaptation for our discussion of (‡). We give a stability result for (‡). We then demonstrate the stability behaviour of some simple quadrature rules applied to an illustrative example equation and we observe that, for a particular family of quadrature rules, the qualitative behaviour of solutions to the example equation is preserved under the discretisation.

Keywords: Convolution Volterra equations; Asymptotic behaviour; Discrete convolution equations

AMS classification: 65R20; 45D05; 45M05

1. Background

Nonlinear Volterra convolution equations of the form:

$$y(t) = g(t) + \int_0^t k(t-s)\varphi(y(s))\,\mathrm{d}s, \qquad t \in \mathbb{R}^+, \tag{1}$$

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have been widely studied (see, for example, [2,7,9]) for their interest in applications (they arise in feedback control theory, etc.). Stability results for the solutions of such equations are given in [2] and also (for example) in [8] which also extends the analysis to certain numerical solutions [1] of the equation. In this work, we also consider a discretised version of (1), which takes the form:

$$y_n = g_n + h \sum_{j=0}^{n} w_{n-j} k_{n-j} \varphi_j, \tag{2}$$

where

$$h > 0, \quad y_n \simeq y(nh), \quad g_n = g(nh), \quad k_n = k(nh), \quad t_n = nh, \quad \varphi_n = \varphi(y_n); \tag{3}$$

we term this the *discrete convolution equation.* In Eq. (2), h is the chosen stepsize, and $\{w_n\}$ are chosen quadrature weights (assumed uniformly bounded) in a family of quadratures

$$\int_0^{nh} k(nh-s)\varphi(y(s))\,\mathrm{d}s \simeq h \sum_{j=0}^{n} w_{n-j} k_{n-j} \varphi(y(jh)).$$

Whilst for many Volterra equations one might employ a quadrature with variable stepsize, there are sound practical reasons for treating convolution equations with a quadrature of the type used here. If $w_0 \neq 0$, then the method will be implicit; Eq. (2) is (for general φ) a nonlinear equation for y_n if $w_0 \neq 0$. In the case of an implicit method, it is assumed that h is chosen so that the discretized equations are uniquely solvable.

By way of example we may use the repeated Euler rule or backward Euler rule, to obtain weights

$$\{w_0 = 0, w_j = 1, j = 1,2,\ldots,\} \quad \text{or} \quad \{w_j = 1, j = 0, 1,\ldots,\},$$

respectively. In practice, we will usually require special starting quadrature weights which do not have a convolution structure. An illustration is provided by the repeated trapezium rule, with

$$w_0 = \tfrac{1}{2}, \qquad w_j = 1, \quad j = 1,2,\ldots,$$

and the ("starting") weight $\frac{1}{2}$ always attached to the term with $j = 0$ in the sum in (2). The class of $\{\rho,\sigma\}$-reducible quadrature rules [1] generates weights of this type, with uniformly bounded weights w_j in the case that the underlying $\{\rho,\sigma\}$ linear multistep formula is strongly-stable. Since the starting weights, if uniformly bounded, affect the analysis only in its detail, we shall restrict ourselves to the idealised situation given by (2).

2. Fundamental results assumed without proof

In the discussion that follows, we shall need to use a number of standard results. For convenience, these are stated below. The notation $\varphi * \psi(t)$ denotes $\int_{-\infty}^{\infty} \varphi(t-s)\psi(s)\,\mathrm{d}s$.

[1] For background to a numerical treatment, see [1].

Theorem 2.1 (Young; see Cotlar and Cignoli [3]). *If the functions $\varphi \in L^p(\mathbb{R})$, and $\psi \in L^q(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, then their convolution $\varphi * \psi \in L^r(\mathbb{R})$, where $1/r = 1/p + 1/q - 1$.*

Theorem 2.2 (Fourier Convolution Theorem). *If the functions $\varphi \in L^1(\mathbb{R})$, and $\psi \in L^1(\mathbb{R})$, then their convolution $\varphi * \psi \in L^1(\mathbb{R})$ satisfies $\widehat{\varphi * \psi} = \widehat{\varphi}\,\widehat{\psi}$, where $\widehat{\psi}(s) = \int_{-\infty}^{\infty} e^{its}\psi(t)\,dt$, is the Fourier transform of the function ψ.*

Theorem 2.3 (Discrete version of Young's Theorem). *If the sequences $\{a_n\} \in \ell^p$ and $\{b_n\} \in \ell^q$, $1 \leqslant p \leqslant \infty$, are such that $1/r = 1/p + 1/q - 1 > 0$ then $\{c_n\} \in \ell^r$, where $\{c_n\}$ is the discrete convolution with $c_n = \Sigma_{j=0}^{n} a_{n-j} b_j$.*

Remark 2.4. This result may be established by applying Theorem 2.1 to the step functions $\varphi(x) = \Sigma_{i=0}^{\infty} a_i I_{[i,(i+1))]}(x), \psi(x) = \Sigma_{i=0}^{\infty} b_i I_{[i,(i+1))]}(x)$ where $I_{[a,b]}(x)$ is the indicator function for the interval $[a, b]$.

Theorem 2.5 (Parseval's formula; see Rudin [10]). *If $\phi \in L^2(\mathbb{R})$, and $\psi \in L^2(\mathbb{R})$ then*

$$\int_{-\infty}^{\infty} \phi(s)\overline{\psi(s)}\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\phi}(t)\overline{\widehat{\psi}(t)}\,dt.$$

Here $\overline{\psi}$ denotes the complex conjugate of the function ψ.

3. An adapted version of Corduneanu's discussion

First (for comparison and motivation of our results in the discrete case) we discuss the behaviour and stability of solutions to

$$y(t) = g(t) + \int_0^t k(t-s)\varphi(y(s))\,ds, \quad t \in \mathbb{R}^+. \tag{4}$$

The result which we discuss here comes from the discussion in Corduneanu [2, pp. 82–90], but, for clarity and convenience later, we provide a reformulation of his proof. We begin by stating the basic theorem [2,Theorem 2.2] for Eq. (4):

Theorem 3.1 (Corduneanu [2]). *Under the assumptions*:

1. *that $g(t), g'(t) \in L^1(\mathbb{R}^+)$ and we define $g(t) = g'(t) = 0$ for $t < 0$,*
2. *that $k(t), k'(t) \in L^1(\mathbb{R}^+)$ and we define $k(t) = k'(t) = 0$ for $t < 0$,*
3. *that $\varphi(\sigma)$ is a continuous bounded function from $\mathbb{R}$ into itself, which satisfies $\sigma\varphi(\sigma) > 0$ for $\sigma \neq 0$,*
4. *that there is a $q \geqslant 0$ such that $\Re\{(1 - isq)\widehat{k}(s)\} \leqslant 0$; $s \in \mathbb{R}$, where $\widehat{k}(s)$ denotes the Fourier transform of $k(t)$,*

any solution $y(t)$ of (1) which is continuous and bounded on $\mathbb{R}^+$ is uniformly continuous and $y(t) \to 0$ as $t \to \infty$.

Our version of the proof of this theorem is given in terms of the following sequence of four lemmas:

Lemma 3.2. *Suppose $y(t)$ is bounded and continuous on $\mathbb{R}^+$ and define*

$$\lambda_{t,q}(\tau) = \int_0^\tau [k(\tau-u) + qk'(\tau-u)]\varphi_t(u)\,\mathrm{d}u + qk(0)\varphi_t(\tau), \quad \tau \geqslant 0,$$

$$\lambda_{t,q}(\tau) = 0, \quad \tau < 0, \ \tau > t.$$

where

$$\varphi_t(\tau) = \varphi(y(\tau)), \quad 0 \leqslant \tau \leqslant t, \qquad \varphi_t(\tau) = 0, \quad \tau < 0, \tau > t.$$

Then, under the assumptions given in Theorem 3.1, $\lambda_{t,q}$ *may be expressed in the form*:

$$\lambda_{t,q}(\tau) = y(\tau) - g(\tau) + qy'(\tau) - qg'(\tau), \quad 0 \leqslant \tau \leqslant t, \tag{5}$$

$$\lambda_{t,q}(\tau) = \int_0^t [k(\tau-u) + qk'(\tau-u)]\varphi(y(u))\,\mathrm{d}u, \quad \tau > t, \tag{6}$$

and

$$\lambda_{t,q} \in L^1 \cap L^2(\mathbb{R}), \quad \widehat{\lambda}_{t,q}(s) = (1 - \mathrm{i}sq)\widehat{k}(s)\widehat{\varphi_t}(s)$$

Proof. By definition,

$$\lambda_{t,q}(\tau) = \int_0^\tau [k(\tau-u) + qk'(\tau-u)]\varphi_t(u)\,\mathrm{d}u + qk(0)\varphi_t(\tau) \tag{7}$$

for $\tau \in \mathbb{R}^+$; consider the equation

$$y(t) = g(t) + \int_0^t k(t-s)\varphi(y(s))\,\mathrm{d}s, \quad t \in \mathbb{R}^+, \tag{8}$$

which, when differentiated, becomes

$$y'(t) = g'(t) + k(0)\varphi(y(t)) + \int_0^t k'(t-s)\varphi(y(s))\,\mathrm{d}s, \quad t \in \mathbb{R}^+. \tag{9}$$

Manipulating (8) and (9), we obtain (5) and (6).

Now, by Young's Theorem, the convolution of $(k + qk') \in L^1(\mathbb{R})$ with $\varphi_t \in L^1 \cap L^2(\mathbb{R})$ is in $L^1 \cap L^2(\mathbb{R})$. Hence, by (7) $\lambda_{t,q} \in L^1 \cap L^2(\mathbb{R})$. Further (since the Fourier transform of $k'(t)$ satisfying assumption 2 is given by $-\mathrm{i}s\hat{k}(s) - k(0)$), it follows from (7) that $\hat{\lambda}_{t,q}(s) = (1 - \mathrm{i}sq)\hat{k}(s)\hat{\varphi}_t(s)$. □

Remark 3.3. Young's Theorem was applied in this context in [5].

Lemma 3.4. *Let q be chosen to satisfy assumption* 4 *in the statement of Theorem* 3.1 *and define* $\rho(t) = \int_0^t \lambda_{t,q}(\tau)\varphi(y(\tau))\,\mathrm{d}\tau$, *then, under the assumptions in Theorem* 3.1,

$$\rho(t) \leqslant 0 \quad \textit{for } t \in \mathbb{R}^+.$$

Proof. We use the notation for φ_t in Lemma 3.2 and $\lambda_t(\tau) = \lambda_{t,q}(\tau)$ for the fixed q given by assumption 4 in Theorem 3.1. With $\rho(t) = \int_0^t \lambda_t(\tau)\varphi(y(\tau))\,d\tau = \int_0^\infty \lambda_t(\tau)\varphi_t(\tau)\,d\tau$, we may apply Parseval's equality for the real-valued function $\rho(t)$ to derive

$$\begin{aligned}\rho(t) &= \int_{-\infty}^{\infty} \lambda_t(s)\overline{\phi_t(s)}\,ds\\ &= (2\pi)^{-1}\int_{-\infty}^{\infty}(1 - isq)\hat{k}(s)\left|\hat{\phi}_t(s)\right|^2\,ds\end{aligned}$$

by Lemma 3.2. Hence

$$\rho(t) = (2\pi)^{-1}\int_{-\infty}^{\infty} \Re\{(1 - isq)\hat{k}(s)\}\,|\varphi_t(s)|^2\,ds,$$

since we know that $\rho(t)$ is a real-valued function. By assumption 4 in the statement of Theorem 3.1, we may conclude that $\rho(t) \leqslant 0$ for $0 \leqslant t < \infty$. □

Lemma 3.5. *Let the assumptions of Theorem* 3.1 *be given and let* $\rho(t)$ *be defined as in Lemma* 3.4, *so that* $\rho(t) \leqslant 0$ *on* $\mathbb{R}^+$. *Then* $0 \leqslant \varphi(y(\tau))y(\tau) \in L^1(\mathbb{R}^+)$ *is uniformly continuous.*

Proof. If we apply the expression (on $0 \leqslant \tau \leqslant t$)

$$\lambda_{t,q}(\tau) = y(\tau) - g(\tau) + qy'(\tau) - qg'(\tau)$$

we may express $\rho(t)$ as

$$\begin{aligned}\rho(t) &= \int_0^t \varphi(y(\tau))\lambda_{t,q}(\tau)\,d\tau\\ &= \int_0^t \varphi(y(\tau))\{y(\tau) - g(\tau) + qy'(\tau) - qg'(\tau)\}\,d\tau\\ &\leqslant 0, \text{ by assumption.}\end{aligned}$$

Hence

$$\int_0^t \varphi(y(\tau))y(\tau)\,d\tau + q\int_{y(0)}^{y(t)} \varphi(y)\,dy \leqslant \Phi\int_0^\infty \{|g(\tau)| + q|g'(\tau)|\}\,d\tau = A,$$

where Φ is chosen so that $|\phi(y)| \leqslant \Phi$ for $y \in \mathbb{R}^+$. By assumption 3 in the statement of Theorem 3.1, the second term in the left-hand side of the inequality satisfies

$$\left|q\int_{y(0)}^{y(t)} \varphi(y)\,dy\right| \leqslant 2q\|\varphi\|_\infty\|y\|_\infty \leqslant B,$$

so we may deduce that:

$$\int_0^t \varphi(y(\tau))y(\tau)\,d\tau \leqslant A + B \quad \text{for } t \in \mathbb{R}^+.$$

This establishes the result that $\varphi(y(\tau))y(\tau) \in L^1(\mathbb{R}^+)$.

The function $\varphi(y(\tau))y(\tau)$ is nonnegative (by assumption 3). It is also uniformly continuous, since φ is continuous and bounded and $y(\tau)$ is uniformly continuous since

$$y'(t) = g'(t) + k(0)\varphi(y(t)) + \int_0^t k'(t-s)\varphi(y(s))\,\mathrm{d}s, \quad t \in \mathbb{R}^+$$

(so that $y'(t)$ is bounded by a constant plus an integrable function). The lemma is established. □

Finally, we require a standard lemma from analysis [2, p. 89]:

Lemma 3.6 (Barbălat). *Let $f(t)$ $(\geqslant 0)$ be both integrable and uniformly continuous on $\mathbb{R}^+$. Then $f(t) \to 0$ as $t \to \infty$.*

Proof of Theorem 3.1. The proof of the theorem now follows easily: We suppose y to be a solution that is continuous and bounded on $\mathbb{R}_+$. Lemmas 3.2 and 3.4 show that $\rho(t)$, defined in Lemma 3.4, satisfies $\rho(t) \leqslant 0$ for $t \geqslant 0$. Lemma 3.5 now shows that $\varphi(y(\tau))y(\tau)$ satisfies the assumptions of Lemma 3.6. By Lemma 3.6,

$$\varphi(y(\tau))y(\tau) \to 0 \qquad \text{as } \tau \to \infty.$$

Let σ be in the ω-set of y, that is, assume there exists a sequence $\{t_i\}_{i\geqslant 0}$ with $t_i \to \infty$, such that $\sigma = \lim_{i\to\infty} y(t_i)$. Such a limit, σ, is finite since y is bounded. Since φ is continuous, and $y(t)$ is continuous,

$$y(t_i)\varphi(y(t_i)) \to \sigma\varphi(\sigma),$$

which must equal 0. By the assumptions on φ, it follows that $\sigma = 0$. Thus 0 is the only point in the ω-set of y. This fact, combined with the fact that y is bounded and uniformly continuous, implies that $\lim_{t\to\infty} y(t) = 0$. □

Remark 3.7. The result given by the above theorem can be strengthened somewhat. It is possible, given the assumption that $y(t)$ is a measurable solution to (1) on $\mathbb{R}^+$, to conclude that $y(t)$ is, as required in the theorem, bounded and uniformly continuous on $\mathbb{R}^+$. We are grateful to a referee who pointed out this mild strengthening of Corduneanu's original result.

We can now interpret our theorem as a stability result for (2):

Corollary 3.8. *Suppose that the function y is a bounded solution to* (1) *and that the function $\tilde{y}$ is a bounded solution to* (1) *with perturbed functions $\tilde{k}$ and $\tilde{g}$ and that $g, \tilde{g}, k, \tilde{k}$ and φ satisfy the assumptions of Theorem* 3.1. *Then $\delta y = y - \tilde{y}$ satisfies $\delta y(t) \to 0$ as $t \to \infty$.*

Equivalently, if y is a bounded solution of (1) for which the assumptions 1 to 4 are satisfied, $\tilde{y}$ is asymptotically stable when the equation suffers perturbations which maintain the validity of the assumptions.

4. A stability result for the discrete convolution equation

In this section, we turn our attention to Eq. (2) which we restate here for convenience:

$$y_n = g_n + h\sum_{j=0}^{n} w_{n-j}k_{n-j}\varphi(y_j). \tag{10}$$

4.1. The Z-transform

We recall the definition and properties (see [4]) of the Z-transform of a sequence $\{a_n\} \equiv \{a_n | n = 0,1,2,\ldots\}$ (all sequences will have subscripts running $0,1,2,\ldots$) as a formal power series

$$Z(\{a_n\})(z) = a_0 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3} + \cdots.$$

In the discussion that follows we shall use the notation $\{[wk]_n\}$ to denote the sequence $\{w_nk_n \mid n = 0,1,2,\ldots\}$ and the notation $\Delta a_n = a_{n+1} - a_n$ and $\Delta[wk]_n = w_{n+1}k_{n+1} - w_nk_n$. Thus,

$$\{\Delta[wk]_n\} = \{(w_{n+1}k_{n+1} - w_nk_n) \mid n = 0,1,2,\ldots\}. \tag{11}$$

4.2. A stability result

By applying methods similar to those discussed for (1) we are able to establish the following theorem:

Theorem 4.1. *With the notation of* (10) *and* (11) *and under the following assumptions*:

1*. $\{g_n\}$, $\{\Delta g_n\} \in \ell^1$;

2*. $\{[wk]_n\}$, $\{\Delta[wk]_n\} \in \ell^1$;

3*. *$h>0$ is fixed and φ is a bounded real-valued function with $|\varphi| \leqslant \Phi$, which satisfies $y\varphi(y) > 0$ for $y \neq 0$;*

4*. *there exists $q \geqslant 0$ such that*

$$\mathrm{Re}\{(h + q[\mathrm{e}^{-\mathrm{i}\theta} - 1])Z(\{[wk]_n\})(\mathrm{e}^{-\mathrm{i}\theta})\} \leqslant 0 \quad \textit{for } \theta \in [0, 2\pi],$$

all solutions of (10) *are bounded and any solution $\{y_n\}$ of* (10) *converges to zero.*

Remark 4.2. We shall write $\|\{a_n\}\|_\infty = \sup_{j\geqslant 0}|a_j|$, $\|\{a_n\}\|_p = (\sum_{j=0}^{\infty}|a_j|^p)^{1/p}$, these quantities being finite for sequences in ℓ^∞ or in ℓ^p, respectively $p > 0$. In particular, $\|\{w_n\}\|_\infty$ is bounded by assumption.

We can interpret the preceding result as a stability result for (10):

Corollary 4.3. *Suppose that $\{y_n\}$ is a solution to* (10) *and that $\{\tilde{y}_n\}$ is a solution to* (10) *with perturbed functions $\{\tilde{k}_n\}$ and $\{\tilde{g}_n\}$ and weights $\{\tilde{w}_j\}$, and that the functions $g, \tilde{g}, k, \tilde{k}$ and φ and weights $\{\tilde{w}_j\}$ satisfy the assumptions of Theorem* 4.1. *Then the sequence $\{\delta y_n\}$ defined by $\delta y_n = y_n - \tilde{y}_n$ satisfies $\delta y_n \to 0$ as $n \to \infty$.*

Remark 4.4. There is a possibility of employing the theory to address the question of whether the numerical solution simulates the behaviour of the analytical solution. If

$$y(nh) = g(nh) + h\sum_{j=0}^{n} w_{n-j}k((n-j)h)\varphi(y(jh)) + \tau_n$$

then we can set $\widetilde{g}_n = g(nh) + \tau_n$ (the functions k, φ and the weights w_n remaining unperturbed), and $\widetilde{y}_n = y(nh)$ to apply the theory to discuss $\lim_{n\to\infty} |y_n - y(nh)|$. This approach focuses attention on deciding whether one can show that the conditions of Theorem 4.1 apply and, in addition, $\{\tau_n\}$, $\{\Delta\tau_n\} \in \ell^1$. Here, the values $\{\tau_n\}$ are local truncation errors associated with the use of the quadrature rule (so that, in fact, $\tau_0 = 0$).

As before, we shall present the proof of the theorem by means of the following sequence of lemmas, which correspond to their counterparts in Section 3, viz. Lemmas 3.2, 3.4, 3.5 and 3.6, respectively.

Lemma 4.5. *Define*

$$\Lambda_j^{(n,q)} = h\sum_{i=0}^{j}\left(w_{j-i}k_{j-i} + \frac{q}{h}\Delta[wk]_{j-i}\right)\varphi_i^{(n)} + qw_0k_0\varphi_{j+1}^{(n)}$$

where

$$\varphi_j^{(n)} = \varphi_j, \quad 0 \leqslant j \leqslant n,$$
$$\varphi_j^{(n)} = 0, \quad j > n.$$

Then under the assumptions given in Theorem 4.1,

(a) *for every* n, q, $\Lambda_j^{(n,q)} \in \ell^1 \cap \ell^2 \equiv \ell^1$;

(b) *for every* n, q,

$$\begin{aligned} Z(\{\Lambda_j^{(n,q)}\})(z) &= \{hZ(w_jk_j)(z) + q(z-1)Z(w_jk_j)(z)\}Z(\varphi_j^{(n)})(z) - qzw_0k_0\varphi_0 \\ &= (h + (z-1)q)Z(w_jk_j)(z)Z(\varphi_j^{(n)})(z) - qzw_0k_0\varphi_0; \end{aligned}$$

(c) $0 \leqslant j \leqslant n$,

$$\Lambda_j^{(n)} = y_j - g_j + \frac{q}{h}(\Delta y_j - \Delta g_j);$$

(d) *for* $j > n$,

$$\Lambda_j^{(n,q)} = h\sum_{i=0}^{n}\left[w_{j-i}k_{j-i} + \frac{q}{h}\Delta[wk]_{j-i}\right]\varphi_i.$$

Remark 4.6. In an attempt to reflect the original continuous analogue of the theorem as closely as possible, we have drawn attention to the fact that $\Lambda_j^{(n,q)} \in \ell^1 \cap \ell^2$; as is well known, $\ell^1 \cap \ell^2 = \ell^1$.

Proof. We may take differences in (10) to give

$$\Delta y_n = \Delta g_n + hw_0k_0\varphi_{n+1} + h\sum_{j=0}^{n}\Delta[wk]_{n-j}\varphi_j.$$

A straightforward substitution in the definition of $\Lambda_j^{(n,q)}$ gives, for $0 \leqslant j \leqslant n$,

$$\Lambda_j^{(n,q)} = y_j - g_j + \frac{q}{h}(\Delta y_j - \Delta g_j).$$

which establishes (c), while, for $j < n$

$$\Lambda_j^{(n,q)} = \sum_{i=0}^{n}[hw_{j-i}k_{j-i} + q\Delta[wk]_{j-i}]\varphi_i,$$

establishing (d).

By Young's theorem, since $\{[wk]_n\} \in \ell^1$ and $\{\Delta[wk]_n\} \in \ell^1$, we have $\{\Lambda_j^{(n,q)}\} \in \ell^1 \cap \ell^2 \equiv \ell^1$ (since, for each fixed n, $\{\varphi_j^{(n)}\}$ is a terminating and a fortiori bounded sequence). This establishes (a).

Finally, (b) follows from (d), since

$$Z(\Lambda_j^{(n,q)})(z) = hZ(wk_j)(z)Z(\varphi_j^{(n)})(z) \\ + q((z-1)Z(wk_j)(z)Z(\varphi_j^{(n)})(z) - z\,w_0k_0\varphi_0). \qquad \square$$

Lemma 4.7. *Define*

$$\rho_{n,q} = \sum_{j=0}^{n}\Lambda_j^{(n,q)}\varphi_j = \sum_{j=0}^{\infty}\Lambda_j^{(n,q)}\varphi_j^{(n)}$$

then by the assumptions in the theorem stated above, $\rho_{n,q} \leqslant 0$ for $n \geqslant 0$.

Proof. Let

$$\begin{aligned}\rho_{n,q} &= \sum_0^{n}\Lambda_j^{(n,q)}\varphi_j = \sum_0^{\infty}\Lambda_j^{(n,q)}\varphi_j^{(n)} \\ &= (2\pi\mathrm{i})^{-1}\int_D Z(\Lambda_j^{(n,q)})\left(\frac{1}{\eta}\right)Z(\varphi_j^{(n)})(\eta)\,\frac{\mathrm{d}\eta}{\eta} \\ &= (2\pi\mathrm{i})^{-1}\int_D \left(h + q\left(\frac{1}{\eta} - 1\right)\right)Z(\{wk_n\})\left(\frac{1}{\eta}\right)\left|Z(\varphi_j^{(n)})(\eta)\right|^2\frac{\mathrm{d}\eta}{\eta} \\ &\quad - (2\pi\mathrm{i})^{-1}\int_D qw_0k_0\varphi_0\frac{1}{\eta}Z(\varphi_j^{(n)})(\eta)\,\frac{\mathrm{d}\eta}{\eta},\end{aligned}$$

by Lemma 4.5, where D is the unit circle, centre at the origin, parametrised by $\eta = \mathrm{e}^{\mathrm{i}\theta}$: $\theta \in [0, 2\pi]$. Now the second integral in this last expression is zero (by Cauchy's residue formula) and we know

ρ_n is real, so we may express ρ_n in the form

$$\rho_{n,q} = (2\pi)^{-1}\int_D \Im\left[\left(h+q\left(\frac{1}{\eta}-1\right)\right)Z(\{wk_n\})\left(\frac{1}{\eta}\right)\right]\left|Z(\varphi_j^{(n)})(\eta)\right|^2\frac{\mathrm{d}\eta}{\eta}$$
$$= (2\pi)^{-1}\int_0^{2\pi}\Re\left[(h+q(\mathrm{e}^{-\mathrm{i}\theta}-1))Z(\{wk_n\})(\mathrm{e}^{-\mathrm{i}\theta})\right]\left|Z(\varphi_j^{(n)})(\mathrm{e}^{\mathrm{i}\theta})\right|^2\mathrm{d}\theta$$
$$\leqslant 0$$

if q is chosen to satisfy assumption 4* in the statement of the theorem. □

Lemma 4.8. *Let $(\rho_{n,q})$ be defined as in Lemma 4.7, so that $\rho_{n,q}\leqslant 0$ for $n\geqslant 0$. Then $\varphi_n y_n \geqslant 0$ and $\{\varphi_n y_n\}\in \ell^1$.*

Proof.

$$\rho_{n,q} = \sum_0^n \varphi_j y_j + \frac{q}{h}\sum_0^n \varphi_j \Delta y_j - \sum_0^n\left(g_j+\frac{q}{h}\Delta g_j\right)\varphi_j \leqslant 0$$

(by Lemma 4.5). So

$$\sum_0^n \varphi_j y_j \leqslant \Phi\sum_0^n\left(|g_j|+\frac{q}{h}|\Delta g_j|+\frac{q}{h}|\Delta y_j|\right)\leqslant A$$
$$= \Phi\left(\|(g_j)\|_l+\frac{q}{h}\|(\Delta g_j)\|_l+2\frac{q}{h}\|(y_j)\|_\infty\right)$$

which is independent of n. Thus $\{\varphi_n y_n\}\in\ell^1$. By assumption 3* in the theorem, $\varphi_n y_n\geqslant 0$ for every $\varphi_j y_j\in\ell^1$ and the result follows. □

Lemma 4.9. *Let f_n be in ℓ^1. Then $f_n\to 0$ as $n\to\infty$.*

Proof. Obvious. □

Proof of Theorem 4.1. With ρ_n defined as in Lemma 4.7, we may deduce from Lemmas 4.5 and 4.7 that $\rho_n\leqslant 0$ under the assumptions of Theorem 4.1. Lemmas 4.8 and Lemma 4.9 then establish that $\varphi_n y_n\to 0$ as $n\to\infty$. We remark, further, that by assumptions 1*, 2* and 3*, $\{y_n\}$ is bounded. Given $\{y_n\}$ is a bounded sequence let $\{x_i\}$ be a convergent subsequence, which converges to c. It follows that $x_i\varphi(x_i)\to c\varphi(c)$ which equals zero by the argument of the preceding paragraph. By assumption 3* in the statement of Theorem 4.1, it follows that $c=0$.

Now let $\varepsilon>0$ be given and define the sequence $\{\widehat{y_m}\}$ to be those elements of the sequence $\{y_n\}$ whose absolute value exceeds ε. Now $\{\widehat{y_m}\}$ must be a finite sequence, since otherwise it is a bounded infinite sequence with limit point 0 (by the above). It follows that $y_n\to 0$ as $n\to\infty$. □

The argument in this section is an analogue of the argument in Section 3 insofar as the continuous case carries over to the discrete case. In particular, if we consider the assumptions of Theorem 3.1

alongside the assumptions for Theorem 4.1, we may make the following observations:

- If g, g' are continuous, then assumption 1 of Theorem 3.1 implies assumption 1* of Theorem 4.1 holds.
- If the quadrature weights are bounded and k, k' are continuous, then assumption 2 of Theorem 3.1 implies assumption 2* of Theorem 4.1 holds.

(We remark that $\|\{\Delta[wk]_n\}\|_1 \leqslant 2\|\{w_n\}\|_\infty \times \|\{k_n\}\|_1$.)

Assumption 3 of Theorem 3.1 is identical to assumption 3* of Theorem 4.1. Accordingly, we shall be concerned to consider the assumptions under which assumption 4 of Theorem 3.1 implies that assumption 4* of Theorem 4.1 holds. We consider this question in the following section in relation to a specific example equation whose kernel satisfies assumption 4 of Theorem 3.1.

5. Application of the theory

We shall review some implications of the discrete theory and then apply it to a simple case. Theorem 3.1 gives a result on stability of solutions to (1) for a wide class of functions φ. As one might expect, the assumptions which are imposed upon the function k to ensure stability for the resulting solution y are correspondingly stringent.

Our assumption for stability of the numerical solution is summarised in the following.

Theorem 5.1. *Assume the functions g, g', k, and k' are continuous, and g, k and φ meet assumptions* 1, 2 *and* 3 *of Theorem* 3.1, *then the condition for stability of the solution to* (2) *is that $h \in \mathcal{H}$ where*

$$\mathcal{H} := \{h \in \mathbb{R} \mid \Re[\mathcal{S}(q;\theta)] \leqslant 0 \ \textit{for all} \ \ \theta \in [0, 2\pi], \ \textit{for some} \ \ q \geqslant 0\}, \tag{12}$$

where

$$\mathcal{S}(q;\theta) = (h + q(e^{-i\theta} - 1))Z(\{wk_n\})(e^{-i\theta}). \tag{13}$$

The set $\mathcal{H}$ in (12) may be interpreted as a *stability interval* for the quadrature method applied to the given convolution equation.

5.1. An illustrative example

In this section, we shall consider (to show how our results can be applied) a class of simple functions k which, under suitable assumptions, satisfy the assumptions underlying Theorem 3.1. Let

$$k(t) = be^{At} \quad \text{for } t \geqslant 0,$$
$$k(t) = 0 \quad \text{for } t < 0.$$

Then for $\Re(A) < 0$, $k(t)$, $k'(t) \in L^1(\mathbb{R}^+)$ (assumption 2 of Theorem 3.1) and for $b < 0, q > 0$, $\Re\{(1 - isq)\hat{k}(s)\} \leqslant 0$ (assumption 4 of Theorem 3.1).

We assume, additionally, that the functions g and φ are chosen to satisfy assumptions 1 and 3 of Theorem 3.1. For a fixed positive value of h, we apply a family of quadrature rules with fixed bounded weights $\{w_n\}$ to solve Eq. (1) with $k(t)$ defined as above. We are particularly interested in whether assumption 4* of Theorem 4.1 is satisfied for the resulting Eq. (2). Assumptions 1*, 2* and 3* of Theorem 4.1 are the discrete analogues of the corresponding assumptions in Theorem 3.1.

For any particular problem we may calculate values of $\Re[\mathcal{S}(q;\theta)]$. Consider the function $k(t)$ defined above with $A = -1$, $b = -1$ and with the weights $\{w_n\}$ chosen from a suitable quadrature rule. If we choose the explicit Euler rule, with $w_0 = 0$, $w_i = 1$ for $i > 0$ then, for each chosen value of q we obtain a range of values of h for which our stability condition holds. We may observe that *for the present equation*, the stability condition is satisfied for any fixed value of $h > 0$ provided a corresponding value of $q > 0$ is chosen sufficiently large.

For the implicit Euler rule, with $w_i = 1$ for every $i \geqslant 0$ we obtain, for the case $q = 0$, that the stability condition given here is satisfied *for all positive values of h*. Thus, $\mathcal{H} \supseteq \mathbb{R}_+$ in both these cases.

6. Summary and further work

Within the present work, we are seeking to reproduce the known stability results for (1) for the discretised version (2). In our treatment, Theorem 4.1 mirrors closely the assumptions and conclusions of Theorem 3.1, and we observed in Section 4 that for g, g', k, k' continuous, assumption 4* of Theorem 4.1 is our stability condition.

In Section 5, we have shown that, for the example equation selected under the implicit Euler rule, the stability assumption is satisfied for all $h > 0$ by choosing $q = 0$. We have further observed that for careful choice of q, we can demonstrate that the stability condition is satisfied for the example equation given in the last subsection, using the explicit Euler rule with any fixed $h > 0$.

The work presented here demonstrates that fruitful avenues of investigation arise when one seeks, in the study of discretized schemes, to mimic analytic properties of functional equations. The work [6] is a mine of such analytical properties. We anticipate that some extensions of the present results to the so-called positive quadratures (those for which the condition

$$\Re(Z(\{w_n\})(z)) \geqslant 0 \quad \text{for } |z| \leqslant 1$$

holds) will be presented in a later note. Also, the requirement that $k(t), k'(t) \in L^1(\mathbb{R}^+)$ is a significant limitation on the class of kernels for which the results contained in this paper apply, but the classes of kernel function which can be analysed may be extended to include kernel functions of the form $k(t) + \xi$ where $k(t)$, $k'(t) \in L^1(\mathbb{R}^+)$ and $\xi > 0$. It would also be of interest to consider equations where φ is unbounded. Indeed, the conditions imposed here, upon k and φ, exclude the classical test equation $y(t) = g(t) + \xi \int_0^t y(s)\,\mathrm{d}s$, which has already received extensive coverage in the literature [1].

An alternative line of approach to that pursued here could be based on Remark 2.4: we might endeavour to base a discrete theory on the direct application of the Corduneanu theory to continuously differentiable functions that approximate step functions of the type presented in this remark. This approach leads to various technical difficulties; it is worth pursuing further, but we doubt that it will lead to any improvement on our stated results.

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References

[1] H. Brunner and P.J. van der Houwen, *The Numerical Solution of Volterra Equations* (North-Holland, Amsterdam, 1986).
[2] C. Corduneanu, *Integral Equations and Stability of Feedback Systems* (Academic Press, New York, 1973).
[3] M. Cotlar and R. Cignoli, *An Introduction to Functional Analysis* (North-Holland, Amsterdam, 1974).
[4] G. Doetsch, *Guide to the Applications of the Laplace and Z-transforms* (Van Nostrand, London, 1971).
[5] N.J. Ford, Convolution integral equations and their discretised versions, M.Sc. Thesis, Victoria University of Manchester, Manchester, UK, 1986.
[6] G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra Integral and Functional Equations* (Cambridge University Press, Cambridge, UK, 1990).
[7] R.C. MacCamy and R.L. Smith, Limits of solutions of nonlinear Volterra equations, *Appl. Anal.* **7** (1977) 19–27.
[8] R.C. MacCamy and P. Weiss, Numerical solution of Volterra integral equations, *Nonlinear Anal.* **3** (5) (1979) 677—695.
[9] R.K. Miller, *Nonlinear Volterra Integral Equations* (Benjamin, New York, 1971).
[10] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1970).